Control of the heat equation with shapes

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Control with constraints

Consider the following control problem: $\Omega \subset \mathbb{R}^N$, bounded, regular,

$$\begin{cases} y_t - \Delta y = u(t,x), \\ y = 0 \text{ on } \partial\Omega, \\ u(t,\cdot) \in \mathcal{U}_c, \quad \forall t \in [0,T]. \end{cases} \tag{1}$$

The control must fulfill a set of constraints given by $\mathcal{U}_c$. For instance, positivity, mass constraint...

Depending on $\mathcal{U}_c$, what are the controllability properties of such a system? When do phenomena such as minimal time appear?

A particular constraint set: 1-shapes

$$\mathcal{U}^1_{\text{shape}} = \{\chi_\omega, \quad |\omega| \leq m_L\}, \quad m_L < |\Omega|.$$

Shape control

We work on "generalised" shapes: let $m_L < |\Omega|$,

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Proposition

Let $m_L < |\Omega|$, $T > 0$, $\varepsilon > 0$. Any nonnegative $y_1 \in L^2(\Omega)$ is ε-approximately reachable from 0 in time T with controls u such that

$$u(t,\cdot) \in \mathcal{U}_{shape} \quad f.a.e\ t \in (0,T).$$

Controllability results with two kinds of controls:

$$u(t,\cdot) = M(t)\chi_{\omega(t)},$$

$$u(t,\cdot) = M\chi_{\omega(t)}.$$

Sharp reachability: comparison principle!

Summary

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Optimal control approach. But $\mathcal{U}_{\text{shape}}$ is not convex!

Relaxation approach

Relaxation approach: we take a relaxed constraint set $\mathcal{U}_r \supset \mathcal{U}_{\text{shape}}$.

- $\mathcal{U}_r$ should be convex.
- Some constraints are preserved, some are not.

With this new *convex* constraint set, and a well-chosen cost C, the optimal control problem

$$\inf_{\substack{u \text{ admissible} \\ u \in \mathcal{U}_r}} C(u)$$

could lead to controls that are actually in the real constraint set $\mathcal{U}_{\text{shape}}$.
cf. Bang-bang property: optimal controls that are actually extremal.

Optimal control problems as convex optimisation problems

We formulate our optimal control problem as a convex optimisation problem.

- Fixed final time $T > 0$, fixed target $y_f \geq 0 \in L^2$.
- Cost C + constraints $\mathcal{U}_r$: TBD!
- $\varepsilon > 0$: **approximate** controllability $\quad \|y_f - y(T)\|_{L^2} \leq \varepsilon$.
- Starting from initial state 0: operator $L_T u = y(T)$.

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$$\inf_{u \in L^2 L^2} C(u) + \delta_r(u) + \delta_{B_\varepsilon(y_f)}(L_T u)$$

$$\delta_r(u) = \begin{cases} 0 & \text{if the control satisfies the constraints f.a.e time} \\ +\infty & \text{otherwise.} \end{cases}$$

$$\delta_{B_\varepsilon(y_f)}(y) = \begin{cases} 0 \text{ if } y \in B_\varepsilon(y_f), \\ +\infty \text{ if } y \notin B_\varepsilon(y_f), \end{cases}, \quad \text{approx. controllability.}$$

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The δ functions

Very generic way of encoding constraints: **indicator functions**.
Strongly singular, enforce extreme *penalization*.

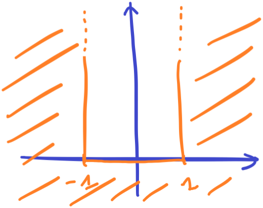

Figure 1: The indicator function $\delta_{[-1,1]}$

Pro: if the optimization problem has solutions, then they correspond to optimal controls for our control problem.
Con: the problem could be ill-defined ($+\infty$ everywhere).

A difficult choice

Two degrees of freedom:

- How do we relax the constraints?
- What cost do we choose?

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Find a "dual problem"?

Summary

Important notions

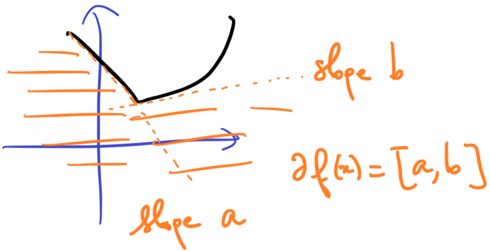
Note that we work with convex functions $E \to \mathbb{R} \cup \{+\infty\}$, with E a Hilbert space.

Convex conjugate:

$$f^*(x) = \sup_{y \in E} \langle x, y \rangle - f(y), \quad \forall x \in E.$$

Subdifferential:

$$\partial f(x) = \{v \in E, \quad f(z) \geq f(x) + \langle v, z - x \rangle, \quad \forall z \in E\}$$



Important properties

Involution: with all the functions we handle,

$$f^{**} = f.$$

A link between subdifferentials and convex conjugation:

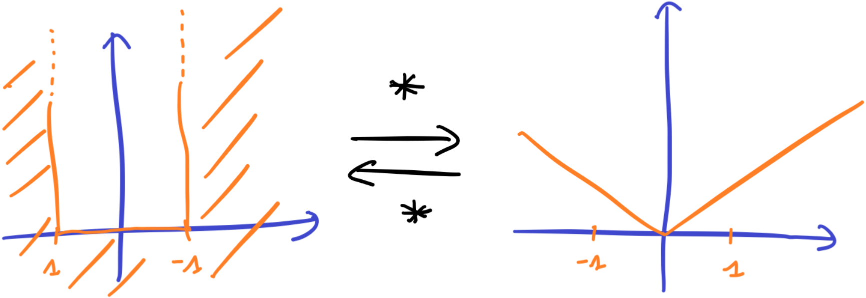
$$\partial f(x) = \{p \in E, \quad \langle p, x \rangle - f(x) = f^*(p)\}$$

Consequence: Legendre-Fenchel identity

$$p \in \partial f(x) \iff x \in \partial f^*(p).$$

An important intuition

An interesting feature of conjugation: growth rates.
Essentially, if f^* has fast growth, f^* has slow growth.



Idea: the dual of a singular problem would have more regularity!

Fenchel-Rockafellar duality

Let E and $\tilde{E}$ be two Hilbert spaces, F and G convex proper functions on E and $\tilde{E}$ resp., and $A \in L(E, \tilde{E})$.

Primal optimization problem: (our optimal control problem)

$$\pi = \inf_{x \in E} F(x) + G(Ax).$$

Using convex conjugation, we can derive its associated **dual problem:**

$$d = \sup_{y \in \tilde{E}} \left(-F^*(A^*y) - G^*(-y)\right) = - \inf_{y \in \tilde{E}} \left(F^*(A^*y) + G^*(-y)\right).$$

Relationship between the values:

Theorem (Fenchel-Rockafellar)

Weak duality $p \geq d$ always holds. Moreover, if there exists $\bar{y} \in \tilde{E}$ such that F^* is continuous at $A^*\bar{y}$ and $G^*(-\bar{y}) < +\infty$, then strong duality holds ie

$$\pi = d \quad \text{and} \quad d \text{ is attained if finite.}$$

Fenchel-Rockafellar duality

Suppose we have strong duality, and the dual problem has a minimizer $y^\star$ (i.e. d is finite and attained). Then, using the assumption,

$$0 \in \partial(F^* \circ A^* + G^* \circ (-Id))(y^\star) = A\partial F^*(A^* y^\star) - \partial G^*(-y^\star).$$

There exists $x^\star \in E$ such that

$$x^\star \in \partial F^*(A^* y^\star), \quad Ax^\star \in \partial G^*(-y^\star),$$

We can "flip" the subdifferentials:

$$A^* y^\star \in \partial F(x^\star), \quad -y^\star \in \partial G(Ax^\star),$$

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$x^\star$ is a minimizer of the primal problem!

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There exists $x^\star \in E$ such that

$$\underbrace{x^\star \in \partial F^*(A^* y^\star)}_{\text{characterization of the control}} \quad , \quad \underbrace{Ax^\star \in \partial G^*(-y^\star)}_{\text{approx controllability}}$$

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The dual approach of relaxation

The **dual problem** can be used to *find minimizers for the primal problem*: **relationship between the solutions.**

Proposition

If there exists $\bar{y} \in E$ *such that* F^* *is continuous at* $A^*\bar{y}$ *and* $G^*(-\bar{y}) < +\infty$, *then if* d *is finite, and attained in* $z^\star \in F$, π *is attained in* $x^\star \in E$ *satisfying*

$$x^\star \in \partial F^*(A^* z^\star), \quad Ax^\star \in \partial G^*(-z^\star). \tag{2}$$

If the **dual problem** satisfies a simple condition and attains its minimum, then we have solved the **primal problem**: we know the *minimum*, and and we have information on *at least one minimizer*.

Parenthesis: HUM

Control system

$$\dot{y} = Ay + Bu.$$

Cost given by $C(u) = \|u\|_{L^2(0,T)}$.
Usually, HUM consists in minimizing the following functional:

$$J_{HUM}(p_T) = \frac{1}{2}\int_0^T \|B^* e^{(T-t)A^*} p_T\|_U^2 dt + \langle y_0, e^{TA^*} p_T\rangle - \langle y_T, p_T\rangle$$

The optimal p_T^* solves:

$$y_T = e^{TA} y_0 + \int_0^T e^{(T-t)A} B \left(\underbrace{B^* e^{(T-t)A^*} p_T^*}_{\text{HUM control}} \right) dt$$

The solution to the dual problem gives the control.

The relaxation approach, revisited

~~How do we relax the constraints and choose the cost?~~

Fenchel-Rockafellar duality: these questions are equivalent to choosing a **dual problem**, and much easier to apprehend in the dual sense.

Primal problem

$$\inf_u F(u) + \delta_{B_\varepsilon(y_f)}(L_T u)$$

Heat equation with source term u →

Convex constraint set

Constrained controllability

At least one $u^\star$ is a shape

Dual problem

$$-\inf_{p_T} F^*(L_T^* p_T) + (\delta_{B_\varepsilon(y_f)})^*(-p_T)$$

Adjoint problem: backward heat equation

Simple condition on F^*

Minimum exists (coercivity...)

$\exists u^\star \in \partial F^*(L_T^* p_T^\star)$

We proceed by choosing the **dual problem**, i.e. the convex function F^* (G^* is given by the approximate controllability problem).

Summary

Choosing the dual problem

Dual problem

$$-\inf_{p_T} F^*(L_T^* p_T) + G^*(-p_T)$$

$$\begin{cases} p_t + \Delta p = 0, \\ p = 0 \quad \text{on } \partial\Omega, \\ p(T, \cdot) = p_T. \end{cases}$$

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- To find shapes: we know that there is at least one optimal control

$$u^\star \in \partial F^*(L_T^* p_T^\star).$$

Subdifferentials and boundaries

Idea: convexify constraints, recover "extremal points" by optimisation.
Define the convex hull of $\mathcal{U}^1_{\text{shape}}$.

$$\mathcal{U}_L := \left\{ u \in L^2(\Omega),\ 0 \le u \le 1 \text{ and } \int_\Omega u \le m_L \right\},$$

$$\partial \delta_{\mathcal{U}_L}(u) = \begin{cases} \{0\} & \text{if } u \in \mathring{\mathcal{U}}_L, \\ \text{a nontrivial cone} & \text{if } u \in \partial \mathcal{U}_L, \\ \emptyset & \text{otherwise.} \end{cases}$$

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Now, for $v \neq 0$, $\quad v \in \partial \delta_{\mathcal{U}_L}(u) \iff u \in \partial \sigma_{\mathcal{U}_L}(v)$.

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Maximisers? Given by the subdifferential!

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The maximisers of $\sup_{u \in \mathcal{U}_L} \langle u, v \rangle$ are on the boundary $\partial \mathcal{U}_L$!

Scalar products and bathtubs

Lemma (relaxed bathtub principle)

Let $v \in L^2$. *Consider the maximisation problem*

$$\sup_{u \in \mathcal{U}_L} \langle u, v \rangle_{L^2}, \quad \mathcal{U}_L := \left\{ u \in L^2(\Omega),\ 0 \leq u \leq 1 \text{ and } \int_\Omega u \leq m_L \right\}.$$

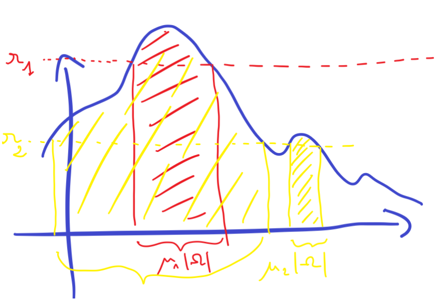
Let $r^\star = \max(0, \inf_{r \in \mathbb{R}} \{ |\{v > r\}| \leq m_L \})$. *Then the maximisers are given by*

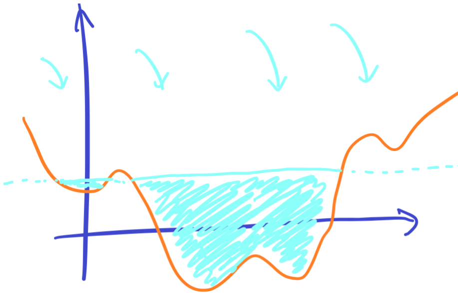
$$u^\star := \chi_{\{v > r^\star\}} + c(x) \chi_{\{v = r^\star\}}, \tag{3}$$

where c *is any measurable function such that* $0 \leq c \leq 1$ *and*

$$\begin{cases} \displaystyle\int_{\{v=r^\star\}} c = m_L - |\{v > r^\star\}| & \text{if } r^\star > 0 \\ \displaystyle\int_{\{v=r^\star\}} c \leq m_L - |\{v > r^\star\}| & \text{if } r^\star = 0 \end{cases}$$

Bathtub principle





The maximisers of the bathtub principle can be characteristic functions:

$$\partial\sigma_{\mathcal{U}_L}(v) = \left\{ u^\star := \chi_{\{v>r^\star\}} + \underline{c(x)\chi_{\{v=r^\star\}}} \right\}.$$

Back to the dual problem

From the *static* bathtub principle to our *dynamic* optimisation problem: how do we choose $F^* : L^2(0,T;L^2(\Omega)) \to \mathbb{R}$?

$$u^\star \in \partial F^* (L_T^* p_T^\star)$$

$$F^* = \qquad \sigma_{\mathcal{U}_L}$$

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$$F^* = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \left(\sigma_{\mathcal{U}_L}(p(t)\right)^2 dt, \\ \dfrac{1}{2}\left(\displaystyle\int_0^T \sigma_{\mathcal{U}_L}(p(t))dt\right)^2 \end{cases}$$

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Quadratic growth does!

Two dual problems

In both cases:

- **Strong duality** holds: F^* continuous in 0, $G^*(0) < \infty$.
- $F^*(L_T^* \cdot) - \langle p_T, y_f \rangle + \varepsilon \|p_T\|_{L^2}$ is coercive **(when** $y_f \geq 0$**)**: **minimum is attained** at $p_T^\star \neq 0 \in L^2$.

Both dual problems will correspond to **optimal control problems**, and these problems have **solutions**.

Do we have shapes? Recall $u^\star \in \partial F^*(L_T^* p_T^\star)$

Technical computation of subdifferentials ($\int_0^T$ and square):

$$F^*(p) = \frac{1}{2}\left(\int_0^T \sigma_{\mathcal{U}_L}(p(t))dt\right)^2$$

$$u^\star(t) \in \left(\int_0^T \sigma_{\mathcal{U}_L}(p(t))dt\right) \partial\sigma_{\mathcal{U}_L}(p^\star(t)).$$

$$F^*(p) = \frac{1}{2}\int_0^T \left(\sigma_{\mathcal{U}_L}(p(t))\right)^2 dt$$

$$u^\star(t) \in \sigma_{\mathcal{U}_L}(p^\star(t))\partial\sigma_{\mathcal{U}_L}(p^\star(t)).$$

Non-empty, contain characteristic functions!

Generality of the approach

What have we used up until now?

- Boundedness of L_T (**well-posedness**).
- For coercivity, some **generic weak convergence properties** of the heat equation.
- The rest is **convex analysis and the bathtub principle**.

Independently of the heat equation, we have proven **approximate controllability to nonnegative states with convexified shapes** (a particular type of *nonnegative controls*) for a very large class of systems!

Shapes for the heat equation

Back to our heat equation problem:

$$u^\star(t) \in \left(\int_0^T \sigma_{\mathcal{U}_L}(p(t))dt\right) \partial\sigma_{\mathcal{U}_L}(p^\star(t)), \quad u^\star(t) \in \sigma_{\mathcal{U}_L}(p^\star(t))\partial\sigma_{\mathcal{U}_L}(p^\star(t)).$$

In both cases we see the object

$$\partial\sigma_{\mathcal{U}_L}(p^\star(t)) = \left\{\chi_{\{p^\star(t)>r^\star(t)\}} + c(x)\chi_{\{p^\star(t)=r^\star(t)\}}\right\}.$$

Shapes for the heat equation

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Heat equation: analytic-hypoelliptic operator! Level sets have **zero measure.**

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Back to our heat equation problem:

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$$\partial\sigma_{\mathcal{U}_L}(p^\star(t)) = \left\{\chi_{\{p^\star(t)>r^\star(t)\}}\right\}.$$

Heat equation: analytic-hypoelliptic operator! Level sets have **zero measure.**
In both cases,

$$u^\star(t) \in \mathcal{U}_{\text{shape}}, \quad f.a.e\ t \in [0, T].$$

Two optimal control problems

Simply compute the conjugates of the F^* we have chosen: we get cost **and** constraints! First problem:

$$F^*(p) = \frac{1}{2}\left(\int_0^T \sigma_{\mathcal{U}_L}(p(t))dt\right)^2$$

$$F(u) = \underbrace{\frac{1}{2}\sup_{t\in[0,T]} \max\left(\|u(t)\|_\infty, \frac{\|u(t)\|_1}{m_L}\right)^2}_{\text{Cost}} + \underbrace{\delta_{u\geq 0}}_{\text{Relaxed constraints}}$$

Second problem:

$$F^*(p) = \frac{1}{2}\int_0^T \left(\sigma_{\mathcal{U}_L}(p(t))\right)^2 dt$$

$$F(u) = \underbrace{\frac{1}{2}\int_0^T \max\left(\|u(t)\|_\infty, \frac{\|u(t)\|_1}{m_L}\right)^2 dt}_{\text{Cost}} + \underbrace{\delta_{u\geq 0}}_{\text{Relaxed constraints}}$$

To sum up

$$\text{Cost } C \xleftarrow{\text{conjugate}} \textbf{Dual problem} \xrightarrow[\text{optimization}]{\text{convex}} p_T^\star \xrightarrow{\text{subdifferentials}} u^\star$$

- The relaxation approach is implemented **indirectly**, by ensuring that strong duality holds and working with the **dual problem**.
- **Note:** to ensure that the dual problem has a minimum, extra (quadratic growth) $\xrightarrow{\text{conjugate}}$ extra relaxation (the penalisation is less brutal).
- This strategy is universal and powerful: general constrained controllability results for a large class of systems.
- For the heat equation: analyticity in space ensures that the controls are shapes (application of the bathtub principle).

Bonus

- General result using convex analysis and FR duality: Hilbert balls are strictly convex → **the optimal control is unique** in both cases.
- For the heat equation: positive minimal control time appears if one restricts the shapes to a subdomain $\omega \subset \Omega$.

Bonus bonus

For the first cost:

$$C(u) = \frac{1}{2} \sup_{t\in[0,T]} \max\left(\|u(t)\|_\infty, \frac{\|u(t)\|_1}{m_L}\right)^2,$$

amplitude of the optimal control does not depend on time, but on the final time T: it solves the minimal norm problem

$$M^*(T) := \inf\{C(u), \quad u \in L^2L^2,\ u \geq 0,\ \|L_T u - y_f\|_2 \leq \varepsilon\},$$

which turns out to be equivalent to the time optimal control problem:

$$T^*(\lambda) = \inf\{T > 0,\ \exists u \in L^2L^2,\ u \geq 0,\ C(u) \leq \lambda,\ \|L_T u - y_f\|_2 \leq \varepsilon\},$$

for $\lambda > 0$.

$$M^*(T) \xrightarrow[T\to+\infty]{} \mu > 0, \quad M^*(T) \xrightarrow[T\to 0]{} +\infty$$

$$M^* \circ T^* = I_{(\mu,+\infty)}, \quad T^* \circ M^* = I_{(0,+\infty)}.$$

Conclusion and remarks

- Approximate controllability of the heat equation *from* 0 *to nonnegative states*, with shapes. In particular, this means any nonnegative state is approx reachable with nonnegative controls!
- Relaxation approach combined with general convex analysis tools: subdifferentials, FR duality.
- Adaptable to other equations, other constraints.
- First cost leads to comprehensive study of a related time-optimal problem.
- FR duality vs. Pontryagin Max Principle?

Thank you!

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